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Appendix A: List of variables in the model

Endogenous	
Variable	*Explanation*
W_A, W_B	Number of warriors warlord A and B each choose to hire
K_A, K_B	Units of capital that warlord A and B each choose to invest
E_A, E_B	Number of natural resource extractors by warlord A and B each choose to hire
K	Total production of goods and services within the economy
Exogenous	
Variable	*Explanation*
N_A, N_B	Population sizes for warlord A and B
Y_A, Y_B	Pre-game/conflict budget allocated to warlord A and B
R_A, R_B	Set of un-extracted natural resources for warlord A and B
c_w	Wage paid to each warrior hired by warlord A and B
c_E	Wage paid to each natural resource extractor hired by warlord A and B
c_k	Price per unit of capital purchased
m	Price paid for each good and service produced and sold
m_R	Price paid on each unit of natural resource that is extracted and sold
σ	Profits earned on the sale of a unit of extracted natural resource by warlord A and B
l_A, l_B	Location of each warlord's territorial stronghold
l_c	Location of the *point of conflict*
Φ	Scalar representing the importance of distance on conflict
α	Mass-effect variable
Includes Both Endogenous and Exogenous Elements	
Variable	*Explanation*
π_A	CSF for warlord A
π_B	CSF for warlord B
V_A, V_B	Income earned by warlord from the production/appropriation of goods and services

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Appendix B: Understanding the contest success function (CSF)

The study of conflict as a theoretical economic problem appears to have originated as its own field through the developing and modeling of a contest. A contest can be defined as a competitive scenario where all players, simultaneously or sequentially, put forth effort to win a prize. A contest can be perfectly discriminating and imperfectly discriminating. A perfectly discriminating contest is such that the player who contributes the most to the conflict wins the prize outright, such as a standard auction.[1] An imperfectly discriminating contest presumes that the player who contributes the highest toward the contest has the greatest probability of winning the prize. Imperfectly discriminating contest models are based upon a contest success function (CSF), or what Hirshleifer calls the technology of conflict.[2] While the functional form of a CSF can vary, all are constructed to relate a player's effort within the contest to his success in said contest. Effort levels enter in the CSF through an impact function which shows the per unit effect of a player's effort on the CSF.

Certain relationships and axioms must be satisfied to properly define a CSF within a contest: probability, marginal effects, anonymity, consistency and independence of irrelevant alternatives.[3] The probability axiom states that the CSF satisfies the properties of a probability distribution; that is, it is assumed the summation of all winning probabilities across all players is equal to one (additive to unity).[4] The marginal effects axiom states that only a player's individual effort levels will increase his probability to win and, hence, one player's effort levels will not help another player's chance of winning.[5] Anonymity requires that each player will have the same probability as an opposing player with the same effort and exogenous factors. In other words, the probability of winning a contest, given a specific effort level and set of exogenous traits, is not dependent upon the actual player himself. Consistency and the independence of irrelevant alternatives both deal specifically with a sub-contest between players. The consistency axiom implies that contests consisting of a smaller number of players will be qualitatively similar to the global contest with a large set of players. Finally, the independence of irrelevant alternatives axiom states that only players active within the contest affect the CSF; that is, the contest should/cannot depend on external players not participating within the contest itself.

Skaperdas (1996) proves that there exists a specific class of CSFs that satisfy the above five axioms. Let a contest take place with $i \in N$ players. Let there exist L types of investments that a player can put efforts toward, where some are allowed to be fixed. Each player is willing to put forth effort levels $\boldsymbol{x}_i \geq 0 \; \forall \; i \in N$, where $\boldsymbol{x}_i \in R_+^L$ is the effort vector of player i for all L investments. Let $\pi_i \colon \boldsymbol{R}_+^{LN} \rightarrow \boldsymbol{R}_+$ be a probability of success where $\boldsymbol{x} \in \boldsymbol{R}_+^{LN}$ is the effort matrix of all N players for all L investments. The above axioms are then satisfied if and only if the CSF has the following form,

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$$(\boldsymbol{b.1})\ \pi_i(\boldsymbol{x}) = \frac{f_i(\boldsymbol{x_i})}{\sum_{j \in N} f_j(\boldsymbol{x_j})}\ \forall\ \boldsymbol{x} \in \boldsymbol{R}_+^{LN}, \forall\ i \in N$$

where $f_i(\cdot)\colon \boldsymbol{R}_+^{LN} \to \boldsymbol{R}_+$ is an *impact function* that is increasing in its arguments.[6] While the above CSF form is required for the five stated axioms to be satisfied, the impact function itself often varies depending on the model and its assumptions.

The two most common explicit impact functions used within the literature are the *ratio* and *difference* models. Originating in the rent-seeking literature, the *ratio* model, first presented by Tullock (1980), equates the impact function as $f(\boldsymbol{x}_i) = \alpha \cdot x_i^\delta$ where $L = 1$, $x_i \in \boldsymbol{R}_+$ is player i's effort put toward the contest, $\alpha_i > 0$ is a positive scalar representing exogenous factors and $\delta > 0$ is commonly known as a *mass effect* factor; that is,

$$(\boldsymbol{b.2})\ \pi_i(\boldsymbol{x}) = \begin{cases} \dfrac{\alpha_i \cdot x_i^\delta}{\sum_{j \in N} \alpha_j \cdot x_j^\delta}\ \forall\ i \in N, & \displaystyle\sum_{j \in N} \alpha_j \cdot x_j^\delta > 0 \\ \dfrac{1}{N}, & \displaystyle\sum_{j \in N} \alpha_j \cdot x_j^\delta \leq 0 \end{cases}$$

The above CSF model is often the workhorse of the conflict theory literature that is also present in other economics fields that incorporate competition in their models such as rent-seeking models, auctions, advertising and sports.[7] The presence of the parameter α_i is meant to represent some exogenous factor that influences a player's probability of winning the contest that does not stem from effort.[8] Such factors are a player's charisma, favoritism or some biased advantage, pre-game standings and so on. The mass effect variable δ is frequently interpreted as an element that captures the marginal increase in a player's probability of winning from an increase in effort. This variable can also be interpreted as a description of the type of conflict taking place. As δ approaches zero, the influence that effort has on the probability of winning the contest becomes less and the contest converges toward a random lottery. As δ approaches infinity, the contest converges toward a perfectly discriminating contest such as an all-pay auction.[9]

The other common, albeit less popular, impact function form is known as the *difference* form (Hirshleifer, 1988; 1989):

$$(\boldsymbol{b.3})\ \pi_i(\boldsymbol{x}) = \frac{1}{1 + \sum_{j \in N \setminus \{i\}} e^{\delta \cdot (\alpha_j \cdot x_j - \alpha_i \cdot x_i)}}\ \forall\ i \in N$$

where e represents the exponential functions and variables α and δ are, again, positive scalars. There are a few notable advantages of the difference CSF form that are generally

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agreed upon. First, from an econometric standpoint, one can easily introduce an additive constant to the impact functions that would normally cause problems for the ratio form; that is, if $f(\boldsymbol{x}_i) = x_i$ and the impact function for each $i \in N$ is $f_i(x_i, c) = x_i + c$, where $c > 0$, then the *difference* CSF form is

$$(\boldsymbol{b.4})\ \pi_i = \frac{1}{1 + \sum_{j \in N} e^{\delta \cdot (x_j - x_i)}}\ \forall\, i \in N,$$

while the *ratio* CSF form is

$$(\boldsymbol{b.5})\ \pi_i = \frac{1}{1 + \dfrac{(J-1) \cdot c + \sum_{j \in N \backslash \{i\}} x_j}{x_i + c}}\ \forall\, i \in N.$$

Second, the functional form of the difference CSF is helpful in creating a decaying effect that territory, environment, climate factors and other causes of fatigue have on a player's effort level. Given the possibilities of imperfect conditions and imperfect information (hidden resources, numerous strongholds and battlefields, clandestine operations and so on), a player may lose a contest yet not lose everything he owns; that is, a player may hide some of his resources so that an opposing player may not win them during a conflict. In other words, the difference CSF form allows a player to not put forth any effort toward the contest and still be able to survive the contest with some nonnegative value. As an example, assume that a contest exists between N players for some exogenous prize and the impact function is again $f(\boldsymbol{x}_i) = x_i$ for all $i \in N$. If the *ratio* CSF is used and player i puts forth zero effort, then

$$(\boldsymbol{b.6})\ \pi_i = \frac{x_i}{x_i + \sum_{j \in N \backslash \{i\}} x_j} = \frac{0}{\sum_{j \in N \backslash \{i\}} x_j} = 0\ \forall\, i \in N.$$

The difference CSF illustrates a different outcome where, again, if player i expends no effort toward the contest, then

$$(\boldsymbol{b.7})\ \pi_i = \frac{1}{1 + \sum_{j \in N \backslash \{i\}} e^{x_j - x_i}} = \frac{1}{1 + \sum_{j \in N \backslash \{i\}} e^{x_j}}\ \forall\, i \in N.$$

The two equations show that the difference CSF allows the opportunity for player i to survive the contest with some portion of the prize when he puts no effort into the contest while the ratio CSF form does not. Therefore, the difference form is said to represent a

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CSF which equates the proportion of the prize one wins and not the probability of winning the entire prize.[10]

Likewise, there are two key problems with the difference CSF. The first and probably most obvious problem that arises with the use of the difference CSF is computation. Given the exponential form of the impact function, models using the difference CSF may not produce a tractable set of results. Secondly, a Nash equilibrium is not always guaranteed, when we seek a Nash Equilibrium in the interior of the strategy space.[11]

Appendix C: Theorems and proofs for the Gates-logit model

Solving maximization problem $(\mathbf{5})$ for warlord A (and, symmetrically, for warlord B), given the Gates-logit formulation of the impact functions described in equation $(\mathbf{3})$, for the choice variables gives rise to the following equilibrium result.

Theorem 1. Let $\hat{l} = \phi \cdot\ (2 \cdot\ l_c - 1)$ and K^* denote the total production of goods and services within the economy. Given assumptions 1 through 3 hold and the following condition is satisfied,

$$(\boldsymbol{c}.\mathbf{1})\ \left(1+\frac{2}{\sqrt{e^{\hat{l}}}}\right) > \frac{\left(N_B+\frac{Y_B}{\sigma}\right)}{\left(N_A+\frac{Y_A}{\sigma}\right)} > \frac{1}{1+2\cdot\sqrt{e^{\hat{l}}}}$$

an interior pure strategy Nash equilibrium exists where:

1. Warlord A and warlord B hire warrior numbers of

$$(\boldsymbol{c}.\mathbf{2})\ W_A^* = \frac{1}{2}\cdot\left(\frac{\sigma}{c_w+\sigma}\right)\cdot\left(\frac{N_A+\frac{Y_A}{\sigma}+N_B+\frac{Y_B}{\sigma}}{1+\frac{1}{\sqrt{e^{\hat{l}}}}}\right);$$

$$(\boldsymbol{c}.\mathbf{3})\ W_B^* = \frac{1}{2}\cdot\left(\frac{\sigma}{c_w+\sigma}\right)\cdot\left(\frac{N_A+\frac{Y_A}{\sigma}+N_B+\frac{Y_B}{\sigma}}{1+\sqrt{\mathrm{e}^{\hat{\mathrm{l}}}}}\right).$$

2. Warlord A and warlord B will invest in capital levels of

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$$(\boldsymbol{c.4})\ K_A^* = \left(\frac{\sigma}{2\cdot c_k}\right)\cdot\left(\frac{\left(1+\frac{2}{\sqrt{e^{\hat{\imath}}}}\right)\cdot\left(N_A+\frac{Y_A}{\sigma}\right)-\left(N_B+\frac{Y_B}{\sigma}\right)}{1+\frac{1}{\sqrt{e^{\hat{\imath}}}}}\right);$$

$$(\boldsymbol{c.5})\ K_B^* = \left(\frac{\sigma}{2\cdot c_k}\right)\cdot\left(\frac{\left(1+2\cdot\sqrt{e^{\hat{\imath}}}\right)\cdot\left(N_B+\frac{Y_B}{\sigma}\right)-\left(N_A+\frac{Y_A}{\sigma}\right)}{1+\sqrt{\mathrm{e}^{\hat{\imath}}}}\right).$$

3. The total production of goods and services within the economy is

$$(\boldsymbol{c.6})\ K^* = K_A^*+K_B^* = \frac{\sigma}{2\cdot c_k}\cdot\left(N_A+\frac{Y_A}{\sigma}+N_B+\frac{Y_B}{\sigma}\right).$$

4. The proportions of the total production of goods and services within the economy are divided as such

$$(\boldsymbol{c.7})\ \pi_{\mathrm{A}} = \frac{1}{1+\sqrt{\mathrm{e}^{\hat{\imath}}}} \text{ and } \pi_B = 1-\pi_{\mathrm{A}} = \frac{1}{1+\frac{1}{\sqrt{e^{\hat{\imath}}}}}.$$

Proof: The proof for Theorem 1 begins by maximizing warlord A and B's optimization problems for the two choice variables W and K. Let λ_A and λ_B be the associated Lagrangian multipliers for maximization problem $(\mathbf{5})$ given the Gates-logit formulation of the impact functions from equation $(\mathbf{3})$. The Lagrangian equations for warlords A and B are

$$(\boldsymbol{c.8})\ \mathcal{L}_A = \left(\frac{m\cdot(K_A+K_B)}{1+\frac{W_B}{W_A}\cdot e^{\hat{\imath}}}\right)+\lambda_A\cdot\left(N_A+\frac{Y_A}{\sigma}-\left(\frac{c_w+\sigma}{\sigma}\right)\cdot W_A-\left(\frac{c_k}{\sigma}\right)\cdot K_A\right);$$

$$(\boldsymbol{c.9})\ \mathcal{L}_B = \left(\frac{m\cdot(K_A+K_B)}{1+\frac{W_A}{W_B}\cdot\frac{1}{e^{\hat{\imath}}}}\right)+\lambda_B\cdot\left(N_B+\frac{Y_B}{\sigma}-\left(\frac{c_w+\sigma}{\sigma}\right)\cdot W_B-\left(\frac{c_k}{\sigma}\right)\cdot K_B\right).$$

Therefore:

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$$(\boldsymbol{c.10})\ \frac{\partial \mathcal{L}_A}{\partial W_A} = 0 \Rightarrow \frac{m \cdot (K_A + K_B) \cdot W_B \cdot e^{\hat{\iota}}}{W_A^2 \cdot \left(1 + \frac{W_B}{W_A} \cdot e^{\hat{\iota}}\right)^2} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot \lambda_A = 0;$$

$$(\boldsymbol{c.11})\ \frac{\partial \mathcal{L}_A}{\partial K_A} = 0 \Rightarrow \frac{m}{1 + \frac{W_B}{W_A} \cdot e^{\hat{\iota}}} - \left(\frac{c_k}{\sigma}\right) \cdot \lambda_A = 0;$$

$$(\boldsymbol{c.12})\ \frac{\partial \mathcal{L}_B}{\partial W_B} = 0 \Rightarrow \frac{m \cdot (K_A + K_B) \cdot W_A \cdot \frac{1}{e^{\hat{\iota}}}}{\left(1 + \frac{W_A}{W_B} \cdot \frac{1}{e^{\hat{\iota}}}\right)^2} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot \lambda_B = 0;$$

$$(\boldsymbol{c.13})\ \frac{\partial \mathcal{L}_B}{\partial K_B} = 0 \Rightarrow \frac{m}{1 + \frac{W_A}{W_B} \cdot \frac{1}{e^{\hat{\iota}}}} - \left(\frac{c_k}{\sigma}\right) \cdot \lambda_B = 0.$$

From equations $(\boldsymbol{c.10})$ and $(\boldsymbol{c.11})$ --- and by rearranging the appropriate variables,

$$(\boldsymbol{c.14})\ (K_A + K_B) \cdot \left(\frac{c_k}{c_w + \sigma}\right) = \frac{W_A^2 \cdot \left(1 + \frac{W_B}{W_A} \cdot e^{\hat{\iota}}\right)}{W_B \cdot e^{\hat{\iota}}}$$

and from equations $(\boldsymbol{c.12})$ and $(\boldsymbol{c.13})$

$$(\boldsymbol{c.15})\ (K_A + K_B) \cdot \left(\frac{c_k}{c_w + \sigma}\right) = \frac{W_B^2 \cdot \left(1 + \frac{W_A}{W_B} \cdot \frac{1}{e^{\hat{\iota}}}\right)}{W_B \cdot \frac{1}{e^{\hat{\iota}}}}.$$

Using the above equations $(\boldsymbol{c.14})$ and $(\boldsymbol{c.15})$, the following relationship is found

$$(\boldsymbol{c.16})\ W_A = W_B \cdot \sqrt{e^{\hat{\iota}}}.$$

By substituting equation $(\boldsymbol{c.16})$ into equation $(\boldsymbol{c.14})$,

$$(\boldsymbol{c.17})\ K_A + K_B = \left(\frac{c_w + \sigma}{c_k}\right) \cdot W_A \cdot \left(1 + \frac{1}{\sqrt{e^{\hat{\iota}}}}\right)\ or\ \left(\frac{c_w + \sigma}{c_k}\right) \cdot W_B \cdot \left(1 + \sqrt{e^{\hat{\iota}}}\right).$$

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The equilibrium level of warriors hired by warlord A is then found by using the budget constraints, described in equation $(\mathbf{1})$ and the relationships stated in equations $(\boldsymbol{c.16})$ and $(\boldsymbol{c.17})$. Warlord B's equilibrium level of warriors is then found by substituting W_A^* into equation $(\boldsymbol{c.16})$. The equilibrium levels of capital investment by warlord A and warlord B are each derived by substituting W_A^* and W_B^*, respectively, into equation $(\boldsymbol{c.17})$.

To show the second-ordered conditions, the bordered Hessian for warlord A is

$$(\boldsymbol{c.18})\ H_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w+\sigma}{\sigma}\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w+\sigma}{\sigma}\right) & \frac{\partial^2 \mathcal{L}_A}{\partial W_A^2} & \frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A} \\ -\frac{c_k}{\sigma} & \frac{\partial^2 \mathcal{L}_A}{\partial K_A W_A} & \frac{\partial^2 \mathcal{L}_A}{\partial K_A^2} \end{pmatrix}$$

where $\frac{\partial^2 \mathcal{L}_A}{\partial K_A^2} = 0$ and is satisfied when the determinant is greater than 0; that is,

$$(\boldsymbol{c.19})\ \left|H_A^{\mathcal{B}}\right| = 2 \cdot \left(\frac{c_w+\sigma}{\sigma}\right) \cdot \left(\frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A}\right) - \left(\frac{c_k}{\sigma}\right) \cdot \frac{\partial^2 \mathcal{L}_A}{\partial W_A^2} > 0.$$

From equation $(\boldsymbol{c.3})$

$$(\boldsymbol{c.20})\ \frac{\partial^2 \mathcal{L}_A}{\partial W_A^2} = -\left(\frac{2 \cdot (c_w+\sigma)}{W_A^2 \cdot \left(1+\sqrt{e^{\hat{l}}}\right)^2}\right) \cdot \left(\frac{m}{c_k \cdot \sqrt{e^{\hat{l}}}}\right) < 0$$

and from equation $(\boldsymbol{c.11})$

$$(\boldsymbol{c.21})\ \frac{\partial^2 \mathcal{L}_A}{\partial K_A W_A} = \frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A} = \frac{m \cdot W_B \cdot e^{\hat{l}}}{W_A^2 \cdot \left(1+\frac{W_B}{W_A} \cdot e^{\hat{l}}\right)^2} > 0.$$

Equations $(\boldsymbol{c.20})$ and $(\boldsymbol{c.21})$ show that $\frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A}$ is positive while $\frac{\partial^2 \mathcal{L}_A}{\partial W_A^2}$ is negative. Therefore, equation $(\boldsymbol{c.19})$ is positive for all values of l_c ranging from 0 to 1.

Equally, the bordered Hessian for warlord B is

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$$(\boldsymbol{c.22})\ H_A^{\mathcal{B}} = \begin{pmatrix} 0 & -\left(\frac{c_w+\sigma}{\sigma}\right) & -\frac{c_k}{\sigma} \\ -\left(\frac{c_w+\sigma}{\sigma}\right) & \frac{\partial^2\mathcal{L}_B}{\partial W_B^2} & \frac{\partial^2\mathcal{L}_B}{\partial W_B K_B} \\ -\frac{c_k}{\sigma} & \frac{\partial^2\mathcal{L}_B}{\partial K_B W_B} & \frac{\partial^2\mathcal{L}_B}{\partial K_B^2} \end{pmatrix}$$

where $\frac{\partial^2\mathcal{L}_B}{\partial K_B^2} = 0$ and is satisfied when the determinant is greater than 0; that is,

$$(\boldsymbol{c.23})\ \left|H_B^{\mathcal{B}}\right| = 2\cdot\left(\frac{c_w+\sigma}{\sigma}\right)\cdot\left(\frac{\partial^2\mathcal{L}_B}{\partial W_B K_B}\right) - \left(\frac{c_k}{\sigma}\right)\cdot\frac{\partial^2\mathcal{L}_B}{\partial W_B^2} > 0.$$

From equation $(\boldsymbol{c.5})$

$$(\boldsymbol{c.24})\ \frac{\partial^2\mathcal{L}_B}{\partial W_B^2} = -\left(\frac{2\cdot(c_w+\sigma)}{W_B^2\cdot\left(1+\frac{1}{\sqrt{e^{\hat{l}}}}\right)^2}\right)\cdot\left(\frac{m\cdot\sqrt{e^{\hat{l}}}}{c_k}\right) < 0$$

and from equation $(\boldsymbol{c.13})$

$$(\boldsymbol{c.25})\ \frac{\partial^2\mathcal{L}_B}{\partial K_B W_B} = \frac{\partial^2\mathcal{L}_B}{\partial W_B K_B} = \frac{m\cdot W_A\cdot\frac{1}{e^{\hat{l}}}}{W_B^2\cdot\left(1+\frac{W_A}{W_B}\cdot\frac{1}{e^{\hat{l}}}\right)^2} > 0.$$

Equations $(\boldsymbol{c.24})$ and $(\boldsymbol{c.25})$ show that $\frac{\partial^2\mathcal{L}_B}{\partial W_B K_B}$ is positive while $\frac{\partial^2\mathcal{L}_B}{\partial W_B^2}$ is negative. Therefore, equation $(\boldsymbol{c.23})$ is positive for all values of l_c ranging from 0 to 1.

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Appendix D: Theorems and proofs for the subtractive model

Solving maximization problem $(\mathbf{5})$, given the subtractive formulation of the impact functions described in equation $(\mathbf{4})$, for the choice variables gives rise to the following equilibrium result.

Theorem 2. Let $\hat{l} = \phi \cdot (2 \cdot l_c - 1)$ and K^* denote the total production of goods and services within the economy. Given assumptions 1 through 3 hold and the following condition is satisfied,

$$(\boldsymbol{d.1}) \begin{cases} \left(N_A + \frac{Y_A}{\sigma} + N_B + \frac{Y_B}{\sigma}\right) \cdot \left(\frac{\sigma}{c_w + \sigma}\right) > \Omega_A > (N_B - N_A) \cdot \left(\frac{\sigma}{c_w + \sigma}\right); \\ \left(N_A + \frac{Y_A}{\sigma} + N_B + \frac{Y_B}{\sigma}\right) \cdot \left(\frac{\sigma}{c_w + \sigma}\right) > \Omega_B > (N_A - N_B) \cdot \left(\frac{\sigma}{c_w + \sigma}\right)' \end{cases}$$

where $\Omega_A = \left(\frac{2}{\alpha}\right) - \phi \cdot (2 \cdot l_c - 1)$ and $\Omega_B = \left(\frac{2}{\alpha}\right) + \phi \cdot (2 \cdot l_c - 1)$, an interior pure strategy Nash equilibrium exists where

1. Warlord A and warlord B hire warrior numbers of

$$(\boldsymbol{d.2})\ W_A^* = \frac{1}{2} \cdot \left(\left(\frac{\sigma}{c_w + \sigma}\right) \cdot \left(N_A + \frac{Y_A}{\sigma} + N_B + \frac{Y_B}{\sigma}\right) - \Omega_A\right) - \frac{1}{\alpha};$$

$$(\boldsymbol{d.3})\ W_B^* = \frac{1}{2} \cdot \left(\left(\frac{\sigma}{c_w + \sigma}\right) \cdot \left(N_A + \frac{Y_A}{\sigma} + N_B + \frac{Y_B}{\sigma}\right) - \Omega_B\right) - \frac{1}{\alpha}.$$

2. Warlord A and warlord B will invest in capital levels of

$$(\boldsymbol{d.4})\ K_A^* = \frac{1}{2 \cdot c_k}\left(\left(N_A + \frac{Y_A}{\sigma}\right) - \left(N_B + \frac{Y_B}{\sigma}\right) + (c_w + \sigma) \cdot \Omega_A\right);$$

$$(\boldsymbol{d.5})\ K_B^* = \frac{1}{2 \cdot c_k}\left(\left(N_B + \frac{Y_B}{\sigma}\right) - \left(N_A + \frac{Y_A}{\sigma}\right) + (c_w + \sigma) \cdot \Omega_B\right).$$

3. The total production of goods and services within the economy is

$$(\boldsymbol{d.6})\ K^* = K_A^* + K_B^* = \frac{2 \cdot (c_w + \sigma)}{\alpha \cdot c_k}.$$

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4. The proportions of the total production of goods and services within the economy are divided as such

$$(\boldsymbol{d.7})\ \pi_A = \pi_B = \frac{1}{2}$$

Proof: The proof for Theorem 2 begins by maximizing warlord A and B's optimization problems for the two choice variables W and K. Let λ_A and λ_B be the associated Lagrangian multipliers for maximization problems (**5**) given the ratio formulation of the impact functions. The Lagrangian equations for warlords A and B are

$$(\boldsymbol{d.8})\ \mathcal{L}_A = \left(\frac{m \cdot (K_A + K_B)}{1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{\imath})}}\right) + \lambda_A \cdot \left(N_A + \frac{Y_A}{\sigma} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot W_A - \left(\frac{c_k}{\sigma}\right) \cdot K_A\right);$$

$$(\boldsymbol{d.9})\ \mathcal{L}_B = \left(\frac{m \cdot (K_A + K_B)}{1 + e^{\alpha \cdot (W_A - W_B - \phi \cdot \hat{\imath})}}\right) + \lambda_B \cdot \left(N_B + \frac{Y_B}{\sigma} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot W_B - \left(\frac{c_k}{\sigma}\right) \cdot K_B\right).$$

Therefore:

$$(\boldsymbol{d.10})\ \frac{\partial \mathcal{L}_A}{\partial W_A} = 0 \Longrightarrow \frac{\alpha \cdot m \cdot (K_A + K_B) \cdot e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{\imath})}}{\left(1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{\imath})}\right)^2} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot \lambda_A = 0;$$

$$(\boldsymbol{d.11})\ \frac{\partial \mathcal{L}_A}{\partial K_A} = 0 \Longrightarrow \frac{m}{1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{\imath})}} - \left(\frac{c_k}{\sigma}\right) \cdot \lambda_A = 0;$$

$$(\boldsymbol{d.12})\ \frac{\partial \mathcal{L}_B}{\partial W_B} = 0 \Longrightarrow \frac{\alpha \cdot m \cdot (K_A + K_B) \cdot e^{\alpha \cdot (W_A - W_B - \phi \cdot \hat{\imath})}}{\left(1 + e^{\alpha \cdot (W_A - W_B - \phi \cdot \hat{\imath})}\right)^2} - \left(\frac{c_w + \sigma}{\sigma}\right) \cdot \lambda_B = 0;$$

$$(\boldsymbol{d.13})\ \frac{\partial \mathcal{L}_B}{\partial K_B} = 0 \Longrightarrow \frac{m}{1 + e^{\alpha \cdot (W_A - W_B - \phi \cdot \hat{\imath})}} - \left(\frac{c_k}{\sigma}\right) \cdot \lambda_B = 0.$$

The proof for Theorem 2 follows in similar fashion as the previous proof for Theorem 1. From equations $(\boldsymbol{d.10})$ and $(\boldsymbol{d.12})$ --- and, again, by rearranging the appropriate variables,

$$(\boldsymbol{d.14})\ W_A = W_B + \phi \cdot \hat{\imath} - \frac{1}{\alpha} \cdot \ln\left(\frac{c_w + \sigma}{\alpha \cdot c_k \cdot (K_A + K_B) - (c_w + \sigma)}\right)$$

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and from equations $(\boldsymbol{d.12})$ and $(\boldsymbol{d.13})$

$$(\boldsymbol{d.15})\ W_B = W_A - \phi \cdot \hat{l} - \frac{1}{\alpha} \cdot \ln\left(\frac{c_w + \sigma}{\alpha \cdot c_k \cdot (K_A + K_B) - (c_w + \sigma)}\right).$$

Using the above equations $(\boldsymbol{d.14})$ and $(\boldsymbol{d.15})$, the following is found

$$(\boldsymbol{d.16})\ K_A + K_B = 2 \cdot \left(\frac{c_w+\sigma}{\alpha \cdot c_k}\right).$$

By substituting equation $(\boldsymbol{d.16})$ into equation $(\boldsymbol{d.14})$,

$$(\boldsymbol{d.17})\ W_A = W_B + \phi \cdot \hat{l}.$$

The equilibrium level of capital investment by warlord A is then found by using the budget constraints, described in equation $(\mathbf{1})$, and the relationships stated in equations $(\boldsymbol{d.16})$ and $(\boldsymbol{d.17})$. Warlord B's equilibrium level of capital purchased is then obtained by substituting K_A^* into equation $(\boldsymbol{d.16})$. The equilibrium number of warriors hired by warlord A and warlord B are each derived by substituting K_A^* and K_B^* into the budget constraint equation $(\mathbf{1})$.

To show the second-ordered conditions, the bordered Hessian for warlord A is the same as equation $(\boldsymbol{c.18})$ where, again, $\frac{\partial^2 \mathcal{L}_A}{\partial K_A^2} = 0$ and is satisfied when the determinant, as seen in equation $(\boldsymbol{c.19})$, is greater than 0. From equation $(\boldsymbol{d.10})$

$$\frac{\partial^2 \mathcal{L}_A}{\partial W_A^2} = \alpha^2 \cdot \left(\frac{\left(e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{l})}\right) \cdot \left(e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{l})} - 1\right)}{\left(1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{l})}\right)^3}\right) \cdot m \cdot (K_A + K_B)$$

which, by applying equation $(\boldsymbol{d.17})$, can be simplified as

$$(\boldsymbol{d.18})\ \frac{\partial^2 \mathcal{L}_A}{\partial W_A^2} = 0.$$

From equation $(\boldsymbol{d.11})$

$$\frac{\partial^2 \mathcal{L}_A}{\partial K_A W_A} = \frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A} = \alpha \cdot m \cdot \frac{e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{l})}}{\left(1 + e^{\alpha \cdot (W_B - W_A + \phi \cdot \hat{l})}\right)^2} > 0,$$

where by again applying equation $(\boldsymbol{d.17})$

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$$(\boldsymbol{d.19})\ \frac{\partial^2 \mathcal{L}_A}{\partial K_A W_A} = \frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A} = \alpha \cdot m \cdot \frac{1}{4} > 0.$$

Equations $(\boldsymbol{d.18})$ and $(\boldsymbol{d.19})$ show that while $\frac{\partial^2 \mathcal{L}_A}{\partial W_A^2}$ is equal to zero, $\frac{\partial^2 \mathcal{L}_A}{\partial W_A K_A}$ is positive. Therefore, equation $(\boldsymbol{c.19})$ is positive for all values of l_c ranging from 0 to 1.

Likewise, the second-ordered conditions, the bordered Hessian for warlord A is the same as equation $(\boldsymbol{c.22})$ where, again, $\frac{\partial^2 \mathcal{L}_B}{\partial K_B^2} = 0$ and is satisfied when the determinant, as seen in equation $(\boldsymbol{c.23})$, is greater than 0. From equation $(\boldsymbol{d.12})$

$$\frac{\partial^2 \mathcal{L}_B}{\partial W_B^2} = \alpha^2 \cdot \left(\frac{\left(e^{\alpha\cdot(W_A - W_B - \phi\cdot\hat{l})}\right)\cdot\left(e^{\alpha\cdot(W_A - W_B - \phi\cdot\hat{l})} - 1\right)}{\left(1 + e^{\alpha\cdot(W_A - W_B - \phi\cdot\hat{l})}\right)^3} \right) \cdot m \cdot (K_A + K_B)$$

which, by applying equation $(\boldsymbol{d.17})$, can be simplified as

$$(\boldsymbol{d.20})\ \frac{\partial^2 \mathcal{L}_B}{\partial W_B^2} = 0.$$

From equation $(\boldsymbol{d.13})$

$$\frac{\partial^2 \mathcal{L}_B}{\partial K_B W_B} = \frac{\partial^2 \mathcal{L}_B}{\partial W_B K_B} = \alpha \cdot m \cdot \frac{e^{\alpha\cdot(W_A - W_B - \phi\cdot\hat{l})}}{\left(1 + e^{\alpha\cdot(W_A - W_B - \phi\cdot\hat{l})}\right)^2} > 0,$$

where by again applying equation $(\boldsymbol{d.17})$

$$(\boldsymbol{d.21})\ \frac{\partial^2 \mathcal{L}_B}{\partial K_B W_B} = \frac{\partial^2 \mathcal{L}_B}{\partial W_B K_B} = \alpha \cdot m \cdot \frac{1}{4} > 0.$$

Equations $(\boldsymbol{d.20})$ and $(\boldsymbol{d.21})$ show that while $\frac{\partial^2 \mathcal{L}_B}{\partial W_B^2}$ is equal to zero, $\frac{\partial^2 \mathcal{L}_B}{\partial W_B K_B}$ is positive. Therefore, equation $(\boldsymbol{c.19})$ is positive for all values of l_c ranging from 0 to 1.

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Appendix E: Proposition proofs

Proof of proposition 1. Let $l_c = \frac{1}{2}$ such that $\hat{l} = \phi \cdot (2 \cdot l_c - 1) = 0$. By applying the assumptions $N_A = N_B = N$ and $Y_A = Y_B = Y$ to equations $(\boldsymbol{c.2})$, $(\boldsymbol{c.3})$, $(\boldsymbol{d.2})$ and $(\boldsymbol{d.3})$, the number of warriors hired by each warlord in equilibrium (under each respective model) are identical. Likewise, apply the symmetry assumption to equations $(\boldsymbol{c.4})$, $(\boldsymbol{c.5})$, $(\boldsymbol{d.4})$ and $(\boldsymbol{d.5})$, the level of capital investment by each warlord in equilibrium under each respective model. That is,

- Gates-logit: $W_A^* = W_B^* = \frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}$ and $K_A^* = K_B^* = \frac{\sigma \cdot N + Y}{2 \cdot c_k}$;
- Subtractive: $W_A^* = W_B^* = \left(\frac{\sigma \cdot N + Y}{c_w + \sigma}\right) - \left(\frac{2}{\alpha}\right)$ and $K_A^* = K_B^* = \frac{c_w + \sigma}{\alpha \cdot c_k}$.

To prove that $\pi_A^* = \pi_B^* = \frac{1}{2}$ for all three models is intuitively deduced by the fact that $W_A^* = W_B^*$ for all three models as well. Specifically, due to $\hat{l} = 0$ when $l_c = \frac{1}{2}$, $e^{\hat{l}} = 1$ and equation $(\boldsymbol{c.7})$ states the CSF of both warlords are equal to one-half. Equation $(\boldsymbol{d.7})$ proves the result for the subtractive model.

Proof of proposition 2. Again, let $l_c = \frac{1}{2}$ such that $\hat{l} = \phi \cdot (2 \cdot l_c - 1) = 0$ and assume symmetry amongst the exogenous resources such that $N_A = N_B = N$ and $Y_A = Y_B = Y$. For the Gates-logit model, equations $(\boldsymbol{c.2})$ and $(\boldsymbol{c.3})$ state that the number of warriors hired by each warlord is

$$(\boldsymbol{e.1})\; W_A^* = \underbrace{\left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right)}_{\mathrm{I}} \cdot \underbrace{\left(\frac{1}{1 + \left(\sqrt{e^{\hat{l}}}\right)^{-1}}\right)}_{\mathrm{II}} \text{ and } W_B^*$$

$$= \underbrace{\left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right)}_{\mathrm{III}} \cdot \underbrace{\left(\frac{1}{1 + \sqrt{e^{\hat{l}}}}\right)}_{IV}.$$

From equation $(\boldsymbol{e.1})$, elements I and III are positive by definition. Therefore it is only required to show the effect a change in l_c has on elements II and IV:

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$$(\boldsymbol{e.2})\ \frac{\partial W_A^*}{\partial l_c} = \left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right) \cdot \left(\frac{\phi \cdot \left(\sqrt{e^{\hat{l}}}\right)^{-1}}{\left(1 + \left(\sqrt{e^{\hat{l}}}\right)^{-1}\right)^2}\right) > 0;$$

$$(\boldsymbol{e.3})\ \frac{\partial W_B^*}{\partial l_c} = \left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right) \cdot \left(\frac{-\phi \cdot \sqrt{e^{\hat{l}}}}{\left(1 + \sqrt{e^{\hat{l}}}\right)^2}\right) < 0.$$

Using equation $(\boldsymbol{e.1})$, the total number of warriors hired is shown to be unaffected by l_c:

$$(\boldsymbol{e.4})\ W^* = \left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right) \cdot \left(\frac{1}{1 + \left(\sqrt{e^{\hat{l}}}\right)^{-1}} + \frac{1}{1 + \sqrt{e^{\hat{l}}}}\right) = \frac{\sigma \cdot N + Y}{c_w + \sigma}.$$

From equations $(\boldsymbol{c.4})$ and $(\boldsymbol{c.5})$

$$(\boldsymbol{e.5})\ K_A^* = \left(\frac{\sigma \cdot N + Y}{2 \cdot c_k}\right) \cdot \left(\frac{2 \cdot \left(\sqrt{e^{\hat{l}}}\right)^{-1}}{1 + \left(\sqrt{e^{\hat{l}}}\right)^{-1}}\right) \text{ and } K_B^* = \left(\frac{\sigma \cdot N + Y}{2 \cdot c_k}\right) \cdot \left(\frac{2 \cdot \sqrt{e^{\hat{l}}}}{1 + \sqrt{e^{\hat{l}}}}\right),$$

where, again, the second elements in both equations are the essential pieces to solve. Taking the partial derivatives and performing the appropriate substitutions results in

$$(\boldsymbol{e.6})\ \frac{\partial K_A^*}{\partial l_c} = -\sqrt{e^{\hat{l}}} \cdot 2 \cdot (\sigma \cdot N + Y) < 0;$$

$$(\boldsymbol{e.7})\ \frac{\partial K_B^*}{\partial l_c} = (\sigma \cdot N + Y) \cdot \left(1 + 2 \cdot \left(1 + \sqrt{e^{\hat{l}}}\right)\right) - \left(1 + 2 \cdot \sqrt{e^{\hat{l}}}\right) \cdot (\sigma \cdot N + Y) > 0.$$

The total production of goods and services production within the economy in the equilibrium can be found in equation $(\boldsymbol{c.6})$ where $\frac{\partial K^*}{\partial l_c} = 0$.

For the subtractive model, equations $(\boldsymbol{d.2})$ and $(\boldsymbol{d.3})$ state that the number of warriors hired by each warlord is

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$$(\boldsymbol{e.8})\ W_A^* = \left(\frac{\sigma \cdot N + Y}{c_w + \sigma}\right) + \phi \cdot \left(l_c - \frac{1}{2}\right) - \frac{1}{\alpha} \text{ and } W_B^* = \left(\frac{\sigma \cdot N + Y}{c_w + \sigma}\right) - \phi \cdot \left(l_c - \frac{1}{2}\right) - \frac{1}{\alpha}.$$

Taking the partial derivatives and performing the appropriate substitutions:

$$(\boldsymbol{e.9})\ \frac{\partial W_A^*}{\partial l_c} = \phi > 0;$$

$$(\boldsymbol{e.10})\ \frac{\partial W_B^*}{\partial l_c} = -\phi < 0.$$

Using equation $(\boldsymbol{e.8})$, the total number of warriors hired is shown to be unaffected by l_c:

$$(\boldsymbol{e.11})\ W^* = 2 \cdot \left(\frac{\sigma \cdot N + Y}{c_w + \sigma}\right) - \frac{2}{\alpha}.$$

From equations $(\boldsymbol{d.4})$ and $(\boldsymbol{d.5})$

$$(\boldsymbol{e.12})\ K_A^* = \left(\frac{c_w + \sigma}{2 \cdot c_k}\right) \cdot \left(\frac{2}{\alpha} - \phi \cdot (2 \cdot l_c - 1)\right) \text{ and } K_B^* = \left(\frac{c_w + \sigma}{2 \cdot c_k}\right) \cdot \left(\frac{2}{\alpha} + \phi \cdot (2 \cdot l_c - 1)\right).$$

By again taking the partial derivatives and performing the appropriate substitutions results in

$$(\boldsymbol{e.13})\ \frac{\partial K_A^*}{\partial l_c} = -2 \cdot l_c < 0;$$

$$(\boldsymbol{e.14})\ \frac{\partial K_B^*}{\partial l_c} = 2 \cdot l_c > 0.$$

The total production of goods and services production within the economy in the equilibrium can be found in equation $(\boldsymbol{d.6})$ where $\frac{\partial K^*}{\partial l_c} = 0$.

Proof of proposition 3. Starting with the Gates-logit model, from equation $(\boldsymbol{c.2})$

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$$(\boldsymbol{e.15})\ \frac{\partial W_A^*}{\partial \phi} = \left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right) \cdot \frac{\left(l_c - \frac{1}{2}\right) \cdot \left(\sqrt{e^{\hat{l}}}\right)^{-1}}{\left(1 + \left(\sqrt{e^{\hat{l}}}\right)^{-1}\right)^2}$$

where $\frac{\partial W_A^*}{\partial \phi}$, by definition, is positive when $l_c > \frac{1}{2}$ and negative when $l_c < \frac{1}{2}$. From equation $(\boldsymbol{c.3})$

$$(\boldsymbol{e.16})\ \frac{\partial W_B^*}{\partial \phi} = \left(\frac{\sigma \cdot N + Y}{2 \cdot (c_w + \sigma)}\right) \cdot \frac{-\left(l_c - \frac{1}{2}\right) \cdot \sqrt{e^{\hat{l}}}}{\left(1 + \sqrt{e^{\hat{l}}}\right)^2}$$

where $\frac{\partial W_B^*}{\partial \phi}$, by definition, is positive when $l_c < \frac{1}{2}$ and negative when $l_c > \frac{1}{2}$. For the subtractive model, from equation $(\boldsymbol{d.2})$

$$(\boldsymbol{e.17})\ \frac{\partial W_A^*}{\partial \phi} = 2 \cdot l_c - 1$$

where $\frac{\partial W_A^*}{\partial \phi}$, again by definition, is positive when $l_c > \frac{1}{2}$ and negative when $l_c < \frac{1}{2}$. From equation $(\boldsymbol{d.3})$

$$(\boldsymbol{e.18})\ \frac{\partial W_B^*}{\partial \phi} = 1 - 2 \cdot l_c$$

where $\frac{\partial W_B^*}{\partial \phi}$ is positive when $l_c < \frac{1}{2}$ and negative when $l_c > \frac{1}{2}$.

Proof of proposition 4. The first aspect of proposition 3 is easily proven by equation $(\boldsymbol{d.7})$; that is, if $\pi_A^* = \pi_B^* = \frac{1}{2}$, then $\frac{\partial \pi_A^*}{\partial l_c} = \frac{\partial \pi_B^*}{\partial l_c} = 0$. The second piece is similarly proven using equation $(\boldsymbol{c.7})$ where,

$$(\boldsymbol{e.19})\ \frac{\partial \pi_A^*}{\partial l_c} = \frac{-2 \cdot \phi \cdot \sqrt{\mathrm{e}^{\hat{\mathrm{l}}}}}{\left(1 + \sqrt{\mathrm{e}^{\hat{\mathrm{l}}}}\right)^2} < 0;$$

and

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$$(\boldsymbol{e.20})\ \frac{\partial \pi_B^*}{\partial l_c} = \frac{2 \cdot \phi \cdot \sqrt{e^{\hat{\imath}}}}{\left(1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}\right)^2} > 0.$$

Proof of proposition 5. Beginning with the individual warlord effects, performing simple comparative statics to equations $(\boldsymbol{c.2})$ and $(\boldsymbol{c.3})$ proves the statements of proposition 4 regarding Gates-logit model and the number of warriors hired by each warlord

$$(\boldsymbol{e.21})\ \frac{\partial W_A^*}{\partial N_A} = \frac{\partial W_A^*}{\partial N_B} = \left(\frac{\sigma}{2 \cdot (c_w + \sigma)}\right) \cdot \left(\frac{1}{1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}}\right) > 0;$$

$$(\boldsymbol{e.22})\ \frac{\partial W_B^*}{\partial N_A} = \frac{\partial W_B^*}{\partial N_B} = \left(\frac{\sigma}{2 \cdot (c_w + \sigma)}\right) \cdot \left(\frac{1}{1 + \sqrt{e^{\hat{\imath}}}}\right) > 0,$$

and performing similar comparative statics using equations $(\boldsymbol{c.4})$ and $(\boldsymbol{c.5})$ proves the statements of proposition 4 regarding Gates-logit model and the total production of goods and services within the economy

$$(\boldsymbol{e.23})\ \frac{\partial K_A^*}{\partial N_A} = \left(\frac{\sigma}{2 \cdot c_k}\right) \cdot \left(\frac{1 + \frac{2}{\sqrt{e^{\hat{\imath}}}}}{1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}}\right) > 0;\ \frac{\partial K_A^*}{\partial N_B} = \left(\frac{\sigma}{2 \cdot c_k}\right) \cdot \left(\frac{-1}{1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}}\right) < 0;$$

$$(\boldsymbol{e.24})\ \frac{\partial K_A^*}{\partial Y_A} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(\frac{1 + \frac{2}{\sqrt{e^{\hat{\imath}}}}}{1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}}\right) > 0;\ \frac{\partial K_A^*}{\partial Y_B} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(\frac{-1}{1 + \frac{1}{\sqrt{e^{\hat{\imath}}}}}\right) < 0;$$

$$(\boldsymbol{e.25})\ \frac{\partial K_B^*}{\partial N_A} = \left(\frac{\sigma}{2 \cdot c_k}\right) \cdot \left(\frac{1 + 2 \cdot \sqrt{e^{\hat{\imath}}}}{1 + \sqrt{e^{\hat{\imath}}}}\right) > 0;\ \frac{\partial K_B^*}{\partial N_B} = \left(\frac{\sigma}{2 \cdot c_k}\right) \cdot \left(\frac{-1}{1 + \sqrt{e^{\hat{\imath}}}}\right) < 0;$$

$$(\boldsymbol{e.26})\ \frac{\partial K_B^*}{\partial Y_A} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(\frac{1 + 2 \cdot \sqrt{e^{\hat{\imath}}}}{1 + \sqrt{e^{\hat{\imath}}}}\right) > 0;\ \frac{\partial K_B^*}{\partial Y_B} = \left(\frac{1}{2 \cdot c_k}\right) \cdot \left(\frac{-1}{1 + \sqrt{e^{\hat{\imath}}}}\right) < 0.$$

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To prove the first statements of proposition 4 for the subtractive model, simple comparative statics are again performed to equations $(\boldsymbol{d.2})$ and $(\boldsymbol{d.3})$

$$(\boldsymbol{e.27})\ \frac{\partial W_A^*}{\partial N_A}=\frac{\partial W_A^*}{\partial N_B}=\frac{\partial W_B^*}{\partial N_A}=\frac{\partial W_B^*}{\partial N_B}=\left(\frac{\sigma}{2\cdot(c_w+\sigma)}\right)>0;$$

$$(\boldsymbol{e.28})\ \frac{\partial W_A^*}{\partial Y_A}=\frac{\partial W_A^*}{\partial Y_B}=\frac{\partial W_B^*}{\partial Y_A}=\frac{\partial W_B^*}{\partial Y_B}=\left(\frac{1}{2\cdot(c_w+\sigma)}\right)>0$$

and using equation $(\boldsymbol{d.2})$ and $(\boldsymbol{d.3})$

$$(\boldsymbol{e.29})\ \frac{\partial K_A^*}{\partial N_A}=\frac{\partial K_B^*}{\partial N_B}=\left(\frac{1}{2\cdot c_k}\right)>0;\ \frac{\partial K_A^*}{\partial N_B}=\frac{\partial K_B^*}{\partial N_A}=\left(\frac{-1}{2\cdot c_k}\right);$$

$$(\boldsymbol{e.30})\ \frac{\partial K_A^*}{\partial Y_A}=\frac{\partial K_B^*}{\partial Y_B}=\left(\frac{\sigma}{2\cdot c_k}\right)>0;\ \frac{\partial K_A^*}{\partial Y_B}=\frac{\partial K_B^*}{\partial Y_A}=\left(\frac{-\sigma}{2\cdot c_k}\right).$$

Finally, the second part of proposition 4 needs to be proven. Using equation $(\boldsymbol{d.6})$, neither population sizes nor pre-existing budgets affect the total production of goods and services within the subtractive model; that is, $\frac{\partial K^*}{\partial N_A}=\frac{\partial K_B}{\partial N_B}=\frac{\partial K^*}{\partial Y_A}=\frac{\partial K^*}{\partial Y_B}=0$. From equation $(\boldsymbol{d.6})$, any increase in either warlord's population size and/or pre-existing budget will result in an increase in total production:

$$(\boldsymbol{e.31})\ \frac{\partial K^*}{\partial N_A}=\frac{\partial K^*}{\partial N_B}=\left(\frac{\sigma}{2\cdot c_k}\right)>0;\ \frac{\partial K^*}{\partial Y_A}=\frac{\partial K^*}{\partial Y_B}=\left(\frac{1}{2\cdot c_k}\right)>0.$$

Notes

1. Moldovanu and Sela (2001).
2. Hirshleifer (1988).
3. See Rai and Sarin (2009) for detailed descriptions.
4. Blavatskyy (2010) presents an axiomatic model of a CSF including the possibility that no player wins and so there is a draw.
5. See Münster (2009) for an extension into complementary effects and group contests.
6. See Skaperdas (1996), Clark and Raiis (1998) and Rai and Sarin (2009) for proofs.
7. See Konrad (2007) for a survey.
8. Clark and Raiis (1998).
9. Jia and Skaperdas (2012).
10. Hirshleifer (1988), Skaperdas (1996).
11. See Garfinkel and Skaperdas (2007) for examples.

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